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Identifying the limiting distribution by a general approach of Stein's method

In memory of Professor Xiru Chen (1934–2005)

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Abstract A general exchange pair approach is developed to identify the limiting distribution for any sequence of random variables, by calculating the conditional mean and the conditional second moments. The error of approximation is also studied. In particular, a Berry-Esseen type bound of $O(n^{-3/4})$ is obtained for the Curie-Weiss model at the critical temperature.

Keywords Stein's method, exchangeable pair, limiting distribution, Berry-Esseen bounds, Curie-Weiss model

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1 Introduction

Let $W := W_n$ be a sequence of random variables of interest. Since the exact distribution of W is usually unknown, it would be interesting to find out the limiting distribution. There are several approaches to solve this problem. The classical method is to calculate the characteristic function, which may not be easy to do. Another approach is to use the Stein method. Stein's method was first introduced by [18] for normal approximation. The method is striking because it can deal with not only independent random variables but also dependent random variables and it can also provide an accuracy of the approximation. Stein's idea and method have been extended to various approximation far beyond the normal approximation, for example, to Poisson approximation by Chen [5], to diffusion approximation by Barbour [1], to Gamma approximation by Luk [15], to multivariate normal approximation by Barbour [1], and Meckes and Meckes [16]. We refer to [6] for a thorough coverage of Stein methods fundamentals and recent developments in both theory and applications. We also refer to [2] for a short survey on Stein's method.

By using the exchangeable pair approach of Stein's method, Chatterjee and Shao [4] provided a concrete method to identify the limiting distribution of W under certain conditions. Let (W, W') be an exchangeable pair and $\Delta = W - W'$. Assume that there exist $\lambda > 0$, measurable functions $g(w)$, $r_1(w)$ and $r_2(w)$ such that

$$\mathrm{E}(\Delta \mid W) = \lambda(g(W) + r_1(W)) \tag{1.1}$$

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and

$$\mathrm{E}(\Delta^2 \mid W) = 2\lambda(1 + r_2(W)). \tag{1.2}$$

Let $G(t) = \int_0^t g(s)ds$, $p(t) = c_0 \exp(-G(t))$, where $c_0 = 1/\int_{-\infty}^{\infty} \mathrm{e}^{-G(t)}dt$. Under some regular assumptions on $g(w)$, Chatterjee and Shao [4] showed that if $\mathrm{E}(|r_1(W)| + |r_2(W)| + |\Delta|^3/\lambda) \to 0$, then $W \xrightarrow{d} Y$, where Y has the probability density function $p(t)$.

Assumption (1.2) implies that the conditional second moment of Δ, given W, satisfies a law of large numbers. However, this may not be true in general. The main purpose of this note is to find the limiting distribution of W without the assuming condition (1.2).

The paper is organized as follows. The next section presents the main results. In Section 3, an application to Curie-Weiss model at the critical temperature is discussed with a Berry-Esseen type bound of $O(n^{-3/4})$. Section 4 provides some basic properties of Stein's equation and solution, while the proof of the main results is postponed to Section 5.

2 Main results

Let (W, W') be an exchangeable pair and $\Delta = W - W'$. Assume that there exist $\lambda > 0$ and measurable functions $g(w), v(w) \geqslant 0, r_1(w)$ and $r_2(w)$ such that

$$\mathrm{E}[\Delta \,|\, W] = \lambda(g(W) + r_1(W)) \tag{2.1}$$

and

$$\mathrm{E}[\Delta^2 \,|\, W] = 2\lambda(v(W) + r_2(W)). \tag{2.2}$$

It is well known that conditional expected value and the conditional second moment of Δ, given W, must be a measurable function of W. So Conditions (2.1) and (2.2) are always satisfied.

Let Y be a random variable with the probability density function

$$p(w) = \frac{1}{c_1 v(w)} \exp(-Q(w)), \quad w \in (a, b), \tag{2.3}$$

where $Q(w) = \int_{w_0}^{w} q(t)dt$, w_0 satisfies $g(w_0) = 0$, $q(t) = g(t)/v(t)$ and c_1 is the normalizing constant. Assume $v(a+)p(a+) = v(b-)p(b-) = 0$.

To present our main results, we first introduce some assumptions on the functions g and v. Assume that

(B1) There exist constants $\alpha \geqslant 1$ and $\beta \geqslant 0$ such that for $w_0 \leqslant x \leqslant y < b$

$$|g(x)| \leqslant \alpha g(y) + \beta, \tag{2.4}$$

and $a < y \leqslant x \leqslant w_0$,

$$|g(x)| \leqslant -\alpha g(y) + \beta. \tag{2.5}$$

(B2) There exists a constant $c_2 \geqslant c_1 \max\{1, \mathrm{E}[v(Y)]\}$ such that the equations

$$c_2 g(x) = 1, \quad c_2 g(x) = v(x) \tag{2.6}$$

have at most one solution on (w_0, b) and the equations

$$c_2 g(x) = -1, \quad c_2 g(x) = -v(x) \tag{2.7}$$

have at most one solution on (a, w_0).

(B3) There exists an interval $[l, u] \subset (a, b)$ such that on (a, w_0), $v(x)$ is non-decreasing or $\inf_{x\in(a,l)} v(x) \geqslant c_3$, on (w_0, b), $v(x)$ is non-increasing or $\inf_{x\in(u,b)} v(x) \geqslant c_3$. Moreover, $\|v'\| = \sup_{x\in(a,b)} |v'(x)| \leqslant c_4$.

(B4) There exists a constant $c_5 \geqslant 1$ such that

$$\max\{1, |g'(y)|\} \min\{c_2, (\alpha + 1)(\alpha + \beta c_2)/|g(y)|\}(|y| + \mathrm{E}|Y| + c_2) \leqslant c_5. \tag{2.8}$$

Remark 2.1. If g and g/v are non-decreasing on (a,b) with $(w-w_0)g(w)\geqslant 0$, then Conditions (B1) and (B2) are satisfied with $\alpha=1$ and $\beta=0$.

We are now ready to identify the limiting distribution of W.

Theorem 2.2. *Let* (W,W') *be an exchangeable pair satisfying* (2.1), (2.2) *and the conditions* (B1)–(B4). *If*

$$\mathrm{E}|r_1(W)|+\mathrm{E}|r_2(W)|+\mathrm{E}|\Delta^3/(\lambda v(W))|\to 0, \tag{2.9}$$

then W *converges to* Y *in distribution.*

The next theorem gives an L_1 bound for the approximation.

Theorem 2.3. *Let* (W,W') *be as defined in Theorem* 2.2. *Then for* $\|h'\|<\infty$, *we have*

$$|\mathrm{E}[h(W)]-\mathrm{E}[h(Y)]|\leqslant C\|h'\|\bigg(\mathrm{E}|r_1(W)|+\mathrm{E}|r_2(W)|+\frac{1}{\lambda}\mathrm{E}[|\Delta|^3/v(W)]\bigg), \tag{2.10}$$

where C *is a finite constant depending on* w_0, $c_1,\ldots,c_5$, α *and* β.

When Δ is bounded, we can also give the Berry-Esseen bound for the approximation. Assume that

$$|\Delta|\leqslant\delta. \tag{2.11}$$

Also assume that

(B5) The interval (a,b) can be partitioned by three parts, I_1,I_2 and I_3 and there exists a constant δ_1 such that

$$\mathrm{E}\sup_{|t|\leqslant\delta}\left|\frac{1}{v(W+t)}\right|\mathbb{1}(W\in I_1\cup I_3)\leqslant\delta_1. \tag{2.12}$$

Moreover, v is absolutely continuous and there exist constants c_6 and c_7 such that

$$\sup_{|t|\leqslant\delta}\sup_{x\in I_2}\left|\frac{1}{v(x+t)}\right|\leqslant c_6,\quad \sup_{|t|\leqslant\delta}\sup_{x\in I_2}|v'(x+t)|\leqslant c_7. \tag{2.13}$$

We have the following Berry-Essen type inequality:

Theorem 2.4. *Assume that* (2.1), (2.2), (2.11) *and Conditions* (B1), (B2), (B4) and (B5) *are satisfied. Then*

$$\begin{aligned}&\sup_{z\in(a,b)}|\mathrm{P}(W\leqslant z)-\mathrm{P}(Y\leqslant z)|\\&\quad\leqslant 2c_2\mathrm{E}|r_1(W)|+\mathrm{E}\left|\frac{r_2(W)}{v(W)}\right|+\frac{2(\alpha+\beta c_2+1)\delta^2\delta_1}{\lambda}+(\|p\|+2c_5/c_2)\delta\\&\qquad+\frac{\delta^3}{\lambda}((\alpha+\beta c_2)c_6^2c_7+c_5c_6/c_2+c_5c_6\mathrm{E}|g(W)|+c_6^2(\alpha+1+\beta c_2)\mathrm{E}|g(W)|).\end{aligned} \tag{2.14}$$

3 Application to Curie-Weiss model

Consider the Curie-Weiss model for n-spins $\sigma=(\sigma_1,\ldots,\sigma_n)\in\{-1,1\}^n$ at temperature T. The joint density function of σ is given by

$$\frac{1}{Z_T}\exp\bigg(\frac{\sum_{1\leqslant i<j\leqslant n}\sigma_i\sigma_j}{Tn}\bigg), \tag{3.1}$$

where Z_T is the normalizing constant. For the critical temperature $T=1$, let

$$W=W(\sigma)=n^{-3/4}\sum_{i=1}^n\sigma_i. \tag{3.2}$$

This is a simple statistical mechanical model of ferromagnetic interaction, also called the Ising model on the complete graph. For a detailed mathematical treatment of this model, we refer to the book by [8].

It was proved by [9–11] that as $n \to \infty$, the law of W converges to the distribution with density proportional to $\mathrm{e}^{-x^4/12}$ and Chatterjee and Shao [4] obtained a Berry-Esseen bound of $O(n^{-1/2})$. For various interesting extensions and refinements of their results, one can refer to [12, 17].

In this section, we shall prove that the Berry-Essen bound can be improved to $O(n^{-3/4})$ when the "limiting distribution" is allowed to depend on n, which in turn also shows that the result obtained by Chatterjee and Shao [4] is optimal.

We first construct W' so that (W, W') is an exchangeable pair. Let I be a uniformly distributed random index over $\{1, \ldots, n\}$. For each i, given $\sigma_j, j \neq i, 1 \leqslant j \leqslant n$, let σ_i' be independent of σ_i and have the same conditional distribution as σ_i. Set $W' = W(\sigma_1, \ldots, \sigma_{I-1}, \sigma_I', \sigma_{I+1}, \ldots, \sigma_n)$. Then (W, W') is an exchangeable pair. The following lemma verifies various conditions in Theorems 2.3 and 2.4.

Lemma 3.1. *Let $\Delta = W - W'$. We have,*

$$\mathrm{E}[\Delta \,|\, W] = \frac{1}{3} n^{-3/2} W^3 - n^{-2} W + O(n^{-5/2})(1 + W^5), \tag{3.3}$$

$$\mathrm{E}[\Delta^2 \,|\, W] = 2n^{-3/2} \max\{1 - n^{-1/2} W^2, n^{-1}\} + O(n^{-5/2})(1 + W^4), \tag{3.4}$$

$$|W - W'| \leqslant 2n^{-3/4} \tag{3.5}$$

and

$$\mathrm{E}|W|^3 \leqslant 15. \tag{3.6}$$

From this lemma, we can choose $\lambda = n^{-3/2}$,

$$g(w) = \frac{w^3}{3} - n^{-1/2} w, \quad v(w) = \max\{1 - n^{-1/2} w^2, n^{-1}\},$$
$$|r_1(w)| \leqslant A n^{-1} |w|^3, \quad |r_2(w)| \leqslant A n^{-1} (1 + w^4),$$

where A is an absolute constant.

Let

$$q(w) = \frac{w^3/3 - n^{-1/2} w}{\max\{1 - n^{-1/2} w^2, n^{-1}\}},$$
$$Q(y) = \int_0^y q(w) dw, \tag{3.7}$$

and Y be a random variable with the probability density function

$$p(y) = \frac{1}{c_1 v(y)} \mathrm{e}^{-Q(y)}, \quad y \in (-\infty, \infty), \tag{3.8}$$

where c_1 is the normalizing constant.

Theorem 3.2. *We have for any absolutely continuous function h with $\|h'\| < \infty$,*

$$|\mathrm{E}[h(W) - h(Y)]| \leqslant C \|h'\| n^{-3/4} \tag{3.9}$$

and

$$\sup_z |\mathrm{P}(W \leqslant z) - \mathrm{P}(Y \leqslant z)| \leqslant C n^{-3/4}, \tag{3.10}$$

where C is an absolute constant.

Remark 3.3. From (3.10) and (3.7), we see that $\mathrm{P}(Y \leqslant z)$ involves a term of order $n^{-1/2}$. This indicates that the error bound of order $O(n^{-1/2})$ given by Chatterjee and Shao [4] is optimal.

Proof. By Theorems 2.3 and 2.4, it suffices to show that Conditions (B1)–(B5) are satisfied. It is easy to see that for all $0 \leqslant x \leqslant y \leqslant n^{1/4}$, $|g(x)| \leqslant g(y)+1$ and for $0 \geqslant x \geqslant y \geqslant -n^{1/4}$, $|g(x)| \leqslant -g(y)+1$. It is also not difficult to verify that Conditions (B2)–(B4) are satisfied.

As for (B5), we choose $I_1 = (-\infty, -n^{1/4}/\sqrt{2}), I_2 = [-n^{1/4}/\sqrt{2}, n^{1/4}/\sqrt{2}]$ and $I_3 = (n^{1/4}/\sqrt{2}, \infty)$. Recall that $v(w) = \max\{1 - n^{-1/2}w^2, n^{-1}\}$, then for $w \in I_1 \cup I_3, |t| \leqslant \delta$, $v(w+t) \geqslant n^{-1}$.

By [3, Proposition 4], for $t \geqslant 0$, $\mathrm{P}(|W| \geqslant t) \leqslant 2\mathrm{e}^{-ct^4}$, where $c > 0$ is an absolute constant, we have by integration by parts,

$$\begin{aligned}
&\mathrm{E}\bigg[\sup_{|t|\leqslant\delta}\bigg|\frac{1}{v(W+t)}\bigg| \cdot \mathbb{1}(n^{-1/2}W^2 > 1/2)\bigg] \\
&\quad\leqslant n\mathrm{P}(W^2 > n^{1/2}/2) + n\int_{n^{1/2}/2}^{\infty} \mathrm{P}(W^2 \geqslant y)dy \\
&\quad\leqslant Cn^{-3/4}.
\end{aligned}$$

Similarly, we can prove that

$$\mathrm{E}\bigg|\frac{1}{v(W)}\bigg| \leqslant C, \quad \mathrm{E}|r_1(W)| \leqslant Cn^{-1}, \quad \mathrm{E}|r_2(W)| \leqslant Cn^{-1} \quad \text{and} \quad \mathrm{E}\bigg|\frac{r_2(W)}{v(W)}\bigg| \leqslant Cn^{-1}$$

for some absolute constant C. One can also check that $\|p\| \leqslant C$ for some constant C. This completes the proof. □

We now turn to the proof of Lemma 3.1.

Proof of Lemma 3.1. Let

$$M = n^{-1}\sum_{i=1}^{n} \sigma_i, \quad M_i = n^{-1}\sum_{j\neq i} \sigma_i.$$

Thus $|M| \leqslant 1$ and $|W| \leqslant n^{-1/4}$.

Let $\mathcal{F}$ be the sigma filed generated by σ. By [4], we have

$$\begin{aligned}
\mathrm{E}[\Delta \,|\, \mathcal{F}] &= n^{-3/4}M - n^{-7/4}\sum_{i=1}^{n} \tanh M_i \\
&= n^{-3/4}M - n^{-7/4}\sum_{i=1}^{n}\bigg(M_i - \frac{M_i^3}{3} + O(1)M_i^5\bigg) \\
&= n^{-3/2}\bigg(\frac{W^3}{3} - n^{-1/2}W + O(n^{-1})(1+|W|^5)\bigg). \qquad (3.11)
\end{aligned}$$

This proves (3.3).

By [4] again, we have

$$\begin{aligned}
\mathrm{E}[\Delta^2 \,|\, \mathcal{F}] &= 2n^{-3/2} - 2n^{-5/2}\sum_{i=1}^{n} \sigma_i \tanh M_i \\
&= 2n^{-3/2} - 2n^{-5/2}\sum_{i=1}^{n} \sigma_i\bigg(M_i - \frac{M_i^3}{3} + O(1)M_i^5\bigg) \\
&= 2n^{-3/2}(1 - n^{-1}W^2 + O(n^{-1}(1+W^4))) \\
&= 2n^{-3/2}\max\{1 - n^{-1}W^2, n^{-1}\} + O(n^{-5/2}(1+W^4)). \qquad (3.12)
\end{aligned}$$

The inequalities (3.5) and (3.6) follows directly from [4]. This completes the proof of Lemma 3.1. □

4 Properties of the Stein solution

In this section, we will recall or prove some basic properties of the Stein solution, mainly following the arguments in [4]. We start with the Stein equation.

Let Y be a random variable with the probability density function (2.3). For any absolutely continuous function f, one can show that

$$\mathrm{E}[v(Y)f'(Y)] = \mathrm{E}[g(Y)f(Y)]. \tag{4.1}$$

For a given measureable function h, let f_h be the solution to the following Stein equation:

$$v(w)f'(w) - g(w)f(w) = h(w) - \mathrm{E}[h(Y)], \quad w \in (a,b). \tag{4.2}$$

It is known that the solution f_h can be expressed as for $w \in (a,b)$,

$$\begin{aligned} f_h(w) &= \frac{1}{v(w)p(w)} \int_a^w (h(t) - \mathrm{E}[h(Y)])p(t)dt \\ &= -\frac{1}{v(w)p(w)} \int_w^b (h(t) - \mathrm{E}[h(Y)])p(t)dt. \end{aligned} \tag{4.3}$$

Equation (4.2) has also been discussed in [7,13,14], where an error bound of certain distances (but not including L_1 distance) of beta distribution approximation were obtained. Theorem 2.3 provides an L_1 bound for general non-normal approximation.

Lemma 4.1. *Assume that* (B4) *is satisfied and* $\delta > 0$ *with* $(c_5/c_2)\delta \leqslant 1/2$. *Then we have*

$$\sup_{|t|\leqslant\delta} |g(w+t) - g(w)| \leqslant 2(c_5/c_2^2)\delta + 2(c_5/c_2)\delta|g(w)|. \tag{4.4}$$

Proof. From (2.8) it follows that

$$|g'(w)| \leqslant (c_5/c_2)(1/c_2 + |g(x)|). \tag{4.5}$$

Thus by the mean value theorem,

$$\begin{aligned} \sup_{|t|\leqslant\delta} |g(w+t) - g(w)| &\leqslant \delta \sup_{|t|\leqslant\delta} |g'(w+t)| \\ &\leqslant \frac{c_5}{c_2}\delta(1/c_2 + |g(w+t)|) \\ &\leqslant (c_5/c_2^2)\delta + (c_5/c_2)\delta|g(w)| + (c_5/c_2)\delta \sup_{|t|\leqslant\delta} |g(w+t) - g(w)| \\ &\leqslant (c_5/c_2^2)\delta + (c_5/c_2)\delta|g(w)| + \frac{1}{2} \sup_{|t|\leqslant\delta} |g(w+t) - g(w)|. \end{aligned}$$

This yields (4.4). □

Lemma 4.2. *Let* $\|h\| < \infty$ *and* f_h *be the solution to the equation* (4.2). *Under Conditions* (B1)–(B4), *we have*

$$\|f_h\| \leqslant 2c_2\|h\|, \quad \|f_h g\| \leqslant 2(\alpha + \beta c_2)\|h\|. \tag{4.6}$$

Proof. Recall that the solution to (4.2) is given by

$$\begin{aligned} f_h(w) &= \frac{1}{v(w)p(w)} \int_a^w (h(t) - \mathrm{E}[h(Y)])p(t)dt \\ &= -\frac{1}{v(w)p(w)} \int_w^b (h(t) - \mathrm{E}[h(Y)])p(t)dt. \end{aligned}$$

For $w \leqslant w_0$, define

$$H(w) = \int_a^w p(t)dt - c_2 v(w)p(w)$$

and thus

$$H'(w) = p(w)(1 + c_2 g(w)).$$

Since $g(w_0) = 0$ and there exists at most one point w_1 such that $c_2 g(w_1) + 1 = 0$, $H(w)$ achieves its maximum at w_0 or a. Note that $H(a) \leqslant 0$ by definition and

$$H(w_0) \leqslant 1 - c_2/c_1 \leqslant 0.$$

This gives $H(w) \leqslant 0$ for $w \in (a, w_0]$ and thus $|f_h(w)| \leqslant c_2\|h\|$. Similarly, we can prove $f_h(w) \leqslant c_2\|h\|$ on (w_0, b).

Next, we will give the bound of $\|f_h g\|$. For $w \in (w_0, b)$, by (2.4),

$$\begin{aligned}
|f_h(w)g(w)| &= |g(w)|\mathrm{e}^{Q(w)} \int_w^b \frac{1}{v(t)} \mathrm{e}^{-Q(t)} dt \\
&\leqslant \alpha \mathrm{e}^{Q(w)} \int_w^b \frac{g(t)}{v(t)} \mathrm{e}^{-Q(t)} dt + \beta \mathrm{e}^{Q(w)} \int_w^b p(t)dt \\
&\leqslant \alpha + \beta c_2.
\end{aligned}$$

Similarly, for $w \in (a, w_0)$,

$$|f_h(w)g(w)| \leqslant \alpha + \beta c_2.$$

This completes the proof. □

Lemma 4.3. *Let f_h be the solution given by* (4.3). *Assume that Conditions* (B1)–(B4) *are satisfied and that h is absolutely continuous with $\|h'\| < \infty$. Then*

$$\|f_h\| \leqslant c_5\|h'\|, \quad \|f_h'\| \leqslant C\|h'\|, \tag{4.7}$$

$$\|f_h g'\| \leqslant c_5\|h'\|, \quad \|f_h' g\| \leqslant C\|h'\|, \tag{4.8}$$

where C is a constant depending on $\alpha, \beta, c_2, c_3, c_4$ and c_5.

Proof. Since h is absolutely continuous,

$$h(y) - \mathrm{E}[h(Y)] = \int_a^y h'(t)F(t)dt - \int_y^b h'(t)(1 - F(t))dt.$$

By (4.3), we have

$$\begin{aligned}
f_h(y)v(y)p(y) &= \int_a^y (h(t) - \mathrm{E}[h(Y)])p(t)dt \\
&= -(1 - F(y)) \int_a^y h'(t)F(t)dt - F(y) \int_y^b h'(t)(1 - F(t))dt.
\end{aligned} \tag{4.9}$$

Hence,

$$|f(y)v(y)p(y)| \leqslant \|h'\| \bigg((1 - F(y)) \int_a^y F(t)dt + F(y) \int_y^b (1 - F(t))dt \bigg). \tag{4.10}$$

From (2.6), we know for $y \in (w_0, b)$,

$$1 - F(y) \leqslant c_2 v(y)p(y) \min\{1, 1/|g(y)|\}.$$

By Condition (B2), we can similarly prove

$$\int_y^b v(t)p(t)dt \leqslant c_2 v(y)p(y), \quad \text{for} \quad y \in (a,b).$$

Note that for $y \in (w_0, b)$,

$$\int_a^y F(s)ds = yF(y) - \int_a^y tp(t)dt. \tag{4.11}$$

Thus,

$$\begin{aligned}
&(1-F(y))\int_a^y F(t)dt + F(y)\int_y^b (1-F(t))dt \\
&\quad \leqslant \min\{c_2, (\alpha+\beta c_2)/|g(y)|\}(|y| + \mathrm{E}|Y|)v(y)p(y) + \int_y^b \min\{c_2, (\alpha+\beta c_2)/|g(t)|\}v(t)p(t)dt \\
&\quad \leqslant \min\{c_2, (\alpha+\beta c_2)/|g(y)|\}(|y| + \mathrm{E}|Y|)v(y)p(y) + \min\{c_2, \alpha(\alpha+\beta c_2)/(|g(y)|-\beta)\}\int_y^b v(t)p(t)dt,
\end{aligned}$$

where we used (2.4) in the last line. When $|g(y)| - \beta \leqslant \alpha^2/c_2 + \alpha\beta$, we have $\min\{c_2, \alpha(\alpha+\beta c_2)/(|g(y)| - \beta)\} = c_2$. Otherwise, $|g(y)| \geqslant (\alpha+1)\beta$ and thus

$$\alpha(\alpha+\beta c_2)/(|g(y)|-\beta) \leqslant (\alpha+1)(\alpha+\beta c_2)/|g(y)|.$$

Hence, we can rewrite these terms above as

$$\begin{aligned}
(1-F(y))&\int_a^y F(t)dt + F(y)\int_y^b (1-F(t))dt \\
&\leqslant \min\{c_2, (\alpha+\beta c_2)/|g(y)|\}(|y| + \mathrm{E}|Y|)v(y)p(y) \\
&\quad + \min\{c_2, (\alpha+1)(\alpha+\beta c_2)/|g(y)|\}c_2 v(y)p(y).
\end{aligned} \tag{4.12}$$

Hence, by (2.8), for $y \in (w_0, b)$,

$$\begin{aligned}
&\max\{1, |g'(y)|\}\left((1-F(y))\int_a^y F(t)dt + F(y)\int_y^b (1-F(t))dt\right) \\
&\quad \leqslant v(y)p(y)\max\{1, g'(y)\}\min\{c_2, (\alpha+1)(\alpha+\beta c_2)/|g(y)|\}(|y| + \mathrm{E}|Y| + c_2) \\
&\quad \leqslant c_5 v(y)p(y).
\end{aligned} \tag{4.13}$$

Similarly, for $y \in (a, w_0)$,

$$\max\{1, |g'(y)|\}\left((1-F(y))\int_a^y F(t)dt + F(y)\int_y^b (1-F(t))dt\right) \leqslant c_5 v(y)p(y). \tag{4.14}$$

By (4.10), (4.13) and (4.14), we have

$$\|f_h\| \leqslant c_5\|h'\|, \quad \|f_h g'\| \leqslant c_5\|h'\|. \tag{4.15}$$

Next, we prove the second inequality of (4.7). By [7], we have

$$|f_h'(y)| \leqslant \|h'\|\left(\frac{\int_a^y F(s)ds|G(y)| + \int_y^b (1-F(s))ds|H(y)|}{v^2(y)p(y)}\right), \tag{4.16}$$

where

$$G(y) = v(y)p(y) - g(y)(1-F(y)), \quad H(y) = v(y)p(y) + g(y)F(y).$$

When $y \in (w_0, b)$ and if v is non-increasing,

$$\begin{aligned}
&\int_y^b (1-F(s))ds \left| 1 + \frac{g(y)F(y)}{v(y)p(y)} \right| \\
&\quad \leqslant c_2 \int_y^b v(s)p(s)ds \left| 1 + \frac{|g(y)|}{v(y)p(y)} \right| \\
&\quad \leqslant c_2(1+\alpha+\beta c_2)v(y)
\end{aligned}$$

and thus

$$\frac{\int_y^b (1-F(s))ds|H(y)|}{v^2(y)p(y)} \leqslant c_2(1+\alpha+\beta c_2). \tag{4.17}$$

By (4.11),

$$\frac{\int_a^y F(s)ds|G(y)|}{v^2(y)p(y)} \leqslant \frac{|y|G(y)}{v^2(y)p(y)} + \frac{\mathrm{E}|Y|G(y)}{v^2(y)p(y)}.$$

Recall that

$$v(y)p(y) = -\int_a^y g(t)p(t)dt = \int_y^b g(t)p(t)dt,$$

then

$$\begin{aligned}
\frac{(|y|+\mathrm{E}|Y|)|G(y)|}{v^2(y)p(y)} &= \frac{(|y|+\mathrm{E}|Y|)\left|\int_y^b (g(t)-g(y))p(t)dt\right|}{v^2(y)p(y)} \\
&= \frac{(|y|+\mathrm{E}|Y|)\left|\int_y^b g'(t)(1-F(t))dt\right|}{v^2(y)p(y)} \\
&\leqslant \frac{c_2(|y|+\mathrm{E}|Y|)\int_y^b |g'(t)|v(t)p(t)dt}{v^2(y)p(y)} \\
&\leqslant \frac{c_2\int_y^b |g'(t)|(|t|+\mathrm{E}|Y|)p(t)dt}{v(y)p(y)} \\
&\leqslant \frac{c_5\int_y^b (|g(t)|+1/c_2)p(t)dt}{v(y)p(y)} \\
&\leqslant \frac{c_5\int_y^b (\alpha g(t)+1/c_2+\beta)p(t)dt}{v(y)p(y)} \\
&\leqslant C,
\end{aligned}$$

where C is a constant depending on c_2, c_5, α and β. Therefore,

$$\sup_{y\in(w_0,b)} \frac{\int_a^y F(s)ds|G(y)| + \int_y^b (1-F(s))ds|H(y)|}{v^2(y)p(y)} \leqslant C.$$

Similarly, we can also prove that if $1/v(y) \leqslant 1/c_3$ we have

$$\sup_{y\in(w_0,b)} \frac{\int_a^y F(s)ds|G(y)| + \int_y^b (1-F(s))ds|H(y)|}{v^2(y)p(y)} \leqslant C.$$

For $y \in (a, w_0)$, if $v(y)$ is non-decreasing or $1/v(y) \leqslant 1/c_3$, we also have

$$\sup_{y\in(a,w_0)} \frac{\int_a^y F(s)ds|G(y)| + \int_y^b (1-F(s))ds|H(y)|}{v^2(y)p(y)} \leqslant C.$$

This proves $\|f_h'\| \leqslant C\|h'\|$.

Finally, we will give the bound of $f_h'g$. From (4.2), we have

$$f_h''v - f_h'g = h' + g'f_h - v'f_h'.$$

Thus

$$(f_h'vp)' = f_h''vp - f_h'gp = h'p + g'f_hp - v'f_h'p$$

and

$$\begin{aligned}|g(x)f_h'(x)v(x)p(x)| &= \left|g(x)\int_a^x (h'p)(t) + (g'f_h)(t) - (v'f_h'p)(t)dt\right| \\ &= \left|g(x)\int_x^b (h'p)(t) + (g'f_hp)(t) - (v'f_h'p)(t)dt\right| \\ &\leqslant C\|h'\||g(x)|\min\{F(x), 1-F(x)\} \\ &\leqslant C\|h'\|v(x)p(x).\end{aligned}$$

This proves $\|f_h'g\| \leqslant C\|h'\|$ where C is a constant. □

Lemma 4.4. *Let f_h be the solution given by* (4.3) *and satisfy the conditions in Lemma* 4.3. *We have*

$$|f_h'(x+t) - f_h'(x)| \leqslant C\|h'\||t|/v(x). \tag{4.18}$$

Proof. Observe that

$$\begin{aligned}&|f_h'(x+t) - f_h'(x)| \\ &= \left|\frac{f_h(x+t)g(x+t)}{v(x+t)} - \frac{f_h(x)g(x)}{v(x)} + \frac{h(x+t) - \mathrm{E}[h(Y)]}{v(x+t)} - \frac{h(x) - \mathrm{E}[h(Y)]}{v(x)}\right| \\ &\leqslant |f_h(x+t)g(x+t) + h(x+t) - \mathrm{E}h(Y)| \times \left|\frac{1}{v(x+t)} - \frac{1}{v(x)}\right| \\ &\quad + \frac{1}{v(x)}|f_h(x+t)g(x+t) - f_h(x)g(x)| + \frac{1}{v(x)}|h(x+t) - h(x)| \\ &=: L_1 + L_2 + L_3.\end{aligned}$$

We next give the bounds of L_1, L_2 and L_3. For L_1,

$$\left|\frac{1}{v(x+t)} - \frac{1}{v(x)}\right| \leqslant \frac{|v(x+t) - v(x)|}{v(x)v(x+t)} \leqslant \frac{c_4|t|}{v(x+t)v(x)}$$

and by (4.6), we have

$$L_1 \leqslant \frac{c_4|f_h'(x+t)||t|}{v(x)} \leqslant C\|h'\||t|/v(x). \tag{4.19}$$

For L_2, it is easy to see

$$L_2 \leqslant \frac{(\|f_hg'\| + \|f_h'g\|)|t|}{v(x)} \leqslant C\|h'\||t|/v(x). \tag{4.20}$$

Finally, for L_3, we have $L_3 \leqslant \|h'\||t|/v(x)$. This proves the lemma. □

5 Proof of main results

Theorem 2.2 is a direct consequence of Theorem 2.3.

Proof of Theorem 2.3. Recall that $\Delta = W - W'$ and observe that

$$\begin{aligned}
0 &= \mathrm{E}[(W-W')(f(W)+f(W'))] \\
&= 2\lambda\mathrm{E}[f(W)g(W)] + 2\lambda\mathrm{E}[f(W)r_1(W)] - \mathrm{E}[\Delta(f(W)-f(W'))] \\
&= 2\mathrm{E}[f(W)\mathrm{E}[\Delta\,|\,W]] - \mathrm{E}\bigg[\Delta\int_{-\Delta}^{0} f'(W+t)dt\bigg] \\
&= 2\lambda\mathrm{E}[f(W)g(W)] + 2\lambda\mathrm{E}[f(W)r_1(W)] - 2\lambda\mathrm{E}\bigg[\int_{-\infty}^{\infty} f'(W+t)\hat{K}(t)dt\bigg],
\end{aligned}$$

where $\hat{K}(t) = \mathrm{E}[\frac{\Delta}{2\lambda}(\mathbb{1}(-\Delta \leqslant t \leqslant 0) - \mathbb{1}(0 \leqslant t \leqslant -\Delta))\,|\,W]$. Therefore,

$$\mathrm{E}[f(W)g(W)] = \mathrm{E}\int_{-\infty}^{\infty} f'(W+t)\hat{K}(t)dt - \mathrm{E}[f(W)r_1(W)]. \tag{5.1}$$

By (2.2), we have

$$\frac{1}{2\lambda}\mathrm{E}[f'(W)\Delta^2] = \mathrm{E}[f'(W)v(W)] + \mathrm{E}[f'(W)r_2(W)]. \tag{5.2}$$

Now for f_h given in (4.3), by (5.1), (5.2), Lemmas 4.3 and 4.4, we have

$$\begin{aligned}
|\mathrm{E}[h(W)] - \mathrm{E}[h(Y)]| &= |\mathrm{E}[v(W)f_h'(W) - g(W)f_h(W)]| \\
&= \bigg|\mathrm{E}[f_h(W)r_1(W)] - \mathrm{E}[f_h'(W)r_2(W)] - \mathrm{E}\int_{-\infty}^{\infty} f'(W+t) - f'(W)dt\bigg| \\
&\leqslant C\|h'\|\bigg(\mathrm{E}|r_1(W)| + \mathrm{E}|r_2(W)| + \mathrm{E}\bigg[\frac{1}{v(W)}\int_{-\infty}^{\infty}|t|\hat{K}(t)dt\bigg]\bigg) \\
&\leqslant C\|h'\|(\mathrm{E}|r_1(W)| + \mathrm{E}|r_2(W)| + \mathrm{E}|\Delta^3/\lambda v(W)|).
\end{aligned} \tag{5.3}$$

This completes the proof. □

Proof of Theorem 2.4. The inequality (2.14) is trivial if $2c_5\delta/c_2 > 1$. We assume that $2c_5\delta/c_2 \leqslant 1$ below. Now let $h(W) = \mathbb{1}(W \leqslant z)$. Assume that $|\Delta| \leqslant \delta$. By (5.1),

$$\begin{aligned}
\mathrm{E}[f(W)g(W) + f(W)r_1(W)] &= \mathrm{E}\bigg[\int_{|t|\leqslant\delta} f'(W+t)\hat{K}(t)dt\bigg] \\
&= \mathrm{E}\bigg[\int_{|t|\leqslant\delta} f(W+t)q(W+t)\hat{K}(t)dt\bigg] \\
&\quad + \mathrm{E}\bigg[\int_{|t|\leqslant\delta}\frac{(h(W+t) - \mathrm{E}[h(Y)])\hat{K}(t)}{v(W+t)}dt\bigg].
\end{aligned} \tag{5.4}$$

Observe that

$$\begin{aligned}
\mathrm{E}\bigg[\int_{|t|\leqslant\delta}\frac{(h(W+t) - \mathrm{E}[h(Y)])\hat{K}(t)}{v(W+t)}dt\bigg] &\geqslant \mathrm{E}\bigg[\bigg(\frac{\mathbb{1}(W \leqslant z-\delta) - \mathrm{P}(Y \leqslant z)}{v(W)}\bigg)\int_{|t|\leqslant\delta}\hat{K}(t)dt\bigg] \\
&\quad - \bigg|\mathrm{E}\bigg[\int_{|t|\leqslant\delta}\bigg(\frac{1}{v(W)} - \frac{1}{v(W+t)}\bigg)\hat{K}(t)dt\bigg]\bigg| \\
&\geqslant \mathrm{P}(W \leqslant z-\delta) - \mathrm{P}(Y \leqslant z) - \mathrm{E}\bigg|\frac{r_2(W)}{v(W)}\bigg| \\
&\quad - \bigg|\mathrm{E}\bigg[\int_{|t|\leqslant\delta}\bigg(\frac{1}{v(W)} - \frac{1}{v(W+t)}\bigg)\hat{K}(t)dt\bigg]\bigg|.
\end{aligned} \tag{5.5}$$

Then we have by (5.4),

$$\mathrm{P}(W \leqslant z-\delta) - \mathrm{P}(Y \leqslant z) \leqslant \mathrm{E}\bigg[\int_{|t|\leqslant\delta}(f(W)q(W) - f(W+t)q(W+t))\hat{K}(t)dt\bigg]$$

$$
\begin{aligned}
&+\bigg|\mathrm{E}\bigg[\int_{|t|\leqslant\delta}\bigg(\frac{1}{v(W)}-\frac{1}{v(W+t)}\bigg)\hat{K}(t)dt\bigg]\bigg| \\
&+2c_2\mathrm{E}|r_1(W)|+\mathrm{E}\bigg|\frac{r_2(W)}{v(W)}\bigg|,
\end{aligned} \tag{5.6}
$$

where we used (4.6) in the last inequality.

We only need to obtain the bounds of the following two terms:

$$
\mathrm{E}\bigg[\int_{|t|\leqslant\delta}\bigg(\frac{1}{v(W)}-\frac{1}{v(W+t)}\bigg)\hat{K}(t)dt\bigg] \tag{5.7}
$$

and

$$
\mathrm{E}\bigg[\int_{|t|\leqslant\delta}(f(W)q(W)-f(W+t)q(W+t))\hat{K}(t)dt\bigg]. \tag{5.8}
$$

For the term (5.7), we have

$$
\begin{aligned}
&\mathrm{E}\bigg[\int_{|t|\leqslant\delta}\bigg(\frac{1}{v(W)}-\frac{1}{v(W+t)}\bigg)\hat{K}(t)dt\bigg] \\
&\leqslant \frac{\delta^2}{\lambda}\mathrm{E}\bigg[\sup_{|t|\leqslant\delta}\bigg|\frac{1}{v(W+t)}\bigg|\mathbb{1}(W\in I_1\cup I_3)\bigg] \\
&\quad+\mathrm{E}\bigg[\int_{|t|\leqslant\delta}\bigg(\frac{1}{v(W)}-\frac{1}{v(W+t)}\bigg)\hat{K}(t)dt\mathbb{1}(W\in I_2)\bigg] \\
&\leqslant \frac{\delta^2\delta_1}{\lambda}+\frac{c_6^2c_7\delta^3}{4\lambda}.
\end{aligned}
$$

As to (5.8), we first find the bound of $\sup_{|t|\leqslant\delta}|\frac{f(W+t)g(W+t)}{v(W+t)}-\frac{f(W)g(W)}{v(W)}|\cdot\mathbb{1}(x\in I_2)$. Note that

$$
\begin{aligned}
&\bigg|\frac{f(W+t)g(W+t)}{v(W+t)}-\frac{f(W)g(W)}{v(W)}\bigg| \\
&\quad\leqslant\bigg|\frac{f(W+t)g(W+t)}{v(W+t)}-\frac{f(W+t)g(W+t)}{v(W)}\bigg| \\
&\quad\quad+\bigg|\frac{f(W+t)g(W+t)}{v(W)}-\frac{f(W+t)g(W)}{v(W)}\bigg| \\
&\quad\quad+\bigg|\frac{f(W+t)g(W)}{v(W)}-\frac{f(W)g(W)}{v(W)}\bigg| \\
&\quad=: J_1+J_2+J_3.
\end{aligned}
$$

Recalling that

$$
\sup_{|t|\leqslant\delta}\sup_{x\in I_2}\bigg|\frac{1}{v(x+t)}\bigg|\leqslant c_6,\quad \sup_{|t|\leqslant\delta}\sup_{x\in I_2}|v'(x+t)|\leqslant c_7,
$$

we have

$$
J_1\leqslant\frac{\|fg\|c_4|t|}{v(W)v(W+t)}\leqslant c_6^2c_7\|fg\||t|.
$$

For J_2, we have for $x\in I_2$ and $c_5\delta/c_2\leqslant 1/2$,

$$
\sup_{|t|\leqslant\delta}|g(x+t)-g(x)|\leqslant 2(c_5/c_2^2)\delta+2(c_5/c_2)\delta|g(w)|,
$$

then $J_2\leqslant 2(c_5c_6/c_2)\|f\|\delta(1/c_2+|g(W)|)$. For J_3,

$$
J_3\leqslant c_6\delta\sup_{|t|\leqslant\delta}|f'(W+t)g(W)|
$$

$$\leqslant c_6^2(\|fg\|+1)\delta|g(W)|.$$

Therefore, recalling that $\|f\|\leqslant 2c_2$, $\|fg\|\leqslant 2(\alpha+\beta c_2)$ we have

$$\begin{aligned}&\mathrm{E}\bigg[\int_{|t|\leqslant\delta}(f(W)q(W)-f(W+t)q(W+t))\hat{K}(t)dt\cdot\mathbb{1}(W\in I_2)\bigg]\\&\quad\leqslant\frac{\delta^3}{\lambda}((\alpha+\beta c_2)c_6^2c_7+c_5c_6/c_2+c_5c_6\mathrm{E}|g(W)|+c_6^2(\alpha+1+\beta c_2)\mathrm{E}|g(W)|).\end{aligned}$$

For (5.8), similarly we have

$$\begin{aligned}&\mathrm{E}\bigg[\int_{|t|\leqslant\delta}(f(W)q(W)-f(W+t)q(W+t))\hat{K}(t)dt\bigg]\\&\quad\leqslant\mathrm{E}\bigg[\int_{|t|\leqslant\delta}|f(W)q(W)-f(W+t)q(W+t)|\hat{K}(t)dt\mathbb{1}(W\in I_1\cup I_3)\bigg]\\&\qquad+\mathrm{E}\bigg[\int_{|t|\leqslant\delta}|f(W)q(W)-f(W+t)q(W+t)|\hat{K}(t)dt\mathbb{1}(W\in I_2)\bigg]\\&\quad\leqslant\frac{\|fg\|\delta^2}{\lambda}\mathrm{E}\bigg[\sup_{|t|\leqslant\delta}\bigg|\frac{1}{v(W+t)}\bigg|\bigg]+\mathrm{E}\bigg[\int_{|t|\leqslant\delta}|f(W)q(W)-f(W+t)q(W+t)|\hat{K}(t)dt\mathbb{1}(W\in I_2)\bigg]\\&\quad\leqslant\frac{2(\alpha+\beta c_2)\delta^2\delta_1}{\lambda}+\frac{\delta^3}{\lambda}((\alpha+\beta c_2)c_6^2c_7+c_5c_6/c_2+c_5c_6\mathrm{E}|g(W)|+c_6^2(\alpha+1+\beta c_2)\mathrm{E}|g(W)|).\end{aligned}$$

Therefore, combining those inequalities, we have

$$\begin{aligned}&\mathrm{P}(W\leqslant z-\delta)-\mathrm{P}(Y\leqslant z)\\&\quad\leqslant 2c_2\mathrm{E}|r_1(W)|+\mathrm{E}\bigg|\frac{r_2(W)}{v(W)}\bigg|+\frac{2(\alpha+\beta c_2+1)\delta^2\delta_1}{\lambda}\\&\qquad+\frac{\delta^3}{\lambda}((\alpha+\beta c_2)c_6^2c_7+c_5c_6/c_2+c_5c_6\mathrm{E}|g(W)|+c_6^2(\alpha+1+\beta c_2)\mathrm{E}|g(W)|).\end{aligned}\tag{5.9}$$

Moreover, $\mathrm{P}(z-\delta\leqslant Y\leqslant z)\leqslant\|p\|\delta$. Hence,

$$\begin{aligned}&\mathrm{P}(W\leqslant z-\delta)-\mathrm{P}(Y\leqslant z-\delta)\\&\quad\leqslant 2c_2\mathrm{E}|r_1(W)|+\mathrm{E}\bigg|\frac{r_2(W)}{v(W)}\bigg|+\frac{2(\alpha+\beta c_2+1)\delta^2\delta_1}{\lambda}+\|p\|\delta\\&\qquad+\frac{\delta^3}{\lambda}((\alpha+\beta c_2)c_6^2c_7+c_5c_6/c_2+c_5c_6\mathrm{E}|g(W)|+c_6^2(\alpha+1+\beta c_2)\mathrm{E}|g(W)|).\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathrm{P}(Y\leqslant z+\delta)-\mathrm{P}(W\leqslant z+\delta)\\&\quad\leqslant 2c_2\mathrm{E}|r_1(W)|+\mathrm{E}\bigg|\frac{r_2(W)}{v(W)}\bigg|+\frac{2(\alpha+\beta c_2+1)\delta^2\delta_1}{\lambda}+\|p\|\delta\\&\qquad+\frac{\delta^3}{\lambda}((\alpha+\beta c_2)c_6^2c_7+c_5c_6/c_2+c_5c_6\mathrm{E}|g(W)|+c_6^2(\alpha+1+\beta c_2)\mathrm{E}|g(W)|).\end{aligned}$$

This completes the proof. □

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