Other random variables

The Gamma Distribution, or Γ-distribution

Definition 1 (Γ-distribution)

A random variable is said to have a gamma distribution with parameter (α, λ), $\lambda > 0$, $\alpha > 0$, if its density function is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} & x \geqslant 0 \\ 0 & x < 0, \end{cases}$$

where $\Gamma(\alpha)$, the gamma function, is defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-y} y^{\alpha-1} dy.$$

Gamma function

Lemma 2

The gamma function has the following property:

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1).$$

Specially, if $\alpha = n$, then

$$\Gamma(n) = (n - 1)!.$$

Mean and variance

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Proposition 3

Let X be a gamma random variable with parameter (α, λ). Then

$$\mathbb{E}[X] = \frac{\alpha}{\lambda}, \quad \mathrm{Var}(X) = \frac{\alpha}{\lambda^2}.$$

Proof.

For the expected value,

$$
\begin{aligned}
\mathbb{E}[X] &= \frac{1}{\Gamma(\alpha)} \int_0^\infty x \cdot \lambda e^{-\lambda x} (\lambda x)^{\alpha-1} dx \\
&= \frac{1}{\lambda \Gamma(\alpha)} \int_0^\infty \lambda e^{-\lambda x} (\lambda x)^{\alpha} dx \\
&= \frac{\Gamma(\alpha+1)}{\lambda \Gamma(\alpha)} \\
&= \frac{\alpha}{\lambda}.
\end{aligned}
$$

■

The Cauchy distribution

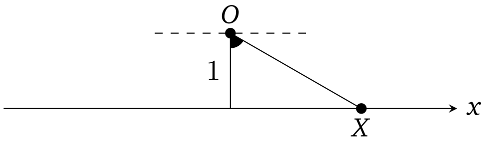
Definition 4

A random variable X is said to have a Cauchy distribution if its density is given by

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

Example 5

Suppose that a narrow-beam flashlight is spun around its center, which is located a unit distance from the x-axis. Consider the point X at which the beam intersects the x-axis when the flashlight has stopped spinning. (If the beam is not pointing toward the x-axis, repeat the experiment.) Find the pdf of X.



Solution.

Let θ be the angle between the flashlight and the y-axis, which is uniformly distributed between $-\pi/2$ and $\pi/2$. The distribution function of X is thus given by

$$\begin{aligned} F(x) &= \mathbb{P}(X \leqslant x) \\ &= \mathbb{P}(\tan\theta \leqslant x) \\ &= \mathbb{P}(\theta \leqslant \tan^{-1} x) \\ &= \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x. \end{aligned}$$

Therefore,

$$f(x) = \frac{d}{dx}F(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty,$$

and hence, X has the Cauchy distribution. ■

The Beta distribution

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Definition 6

A random variable is said to have a beta distribution if its density is given by

$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Remark

The beta distribution can be used to model a random phenomenon whose set of possible values is $[0,1]$. When $a = b$, then the beta distribution is symmetric about $1/2$. When $a = b = 1$, then the beta distribution is identical to $U(0,1)$.

Mean and variance

Proposition 7

If $X \sim \text{Beta}(a, b)$, then

$$\mathbb{E}[X] = \frac{a}{a+b}, \quad \text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}.$$

- Functions of continuous random variables, where the function is monotone.
- Linear combination of random variables: general Uniform distribution, general normal distribution.
- Log-normal distribution.
- Chi-squared distribution.
- Cauchy distribution: $\tan U$ where $U \sim U(-\pi/2, \pi/2)$.
- Inverse distribution function, and generating random numbers.

Distribution of a function of a random variable

- Suppose that we have known the distribution of a continuous random variable X, ...
- and we want to find the distribution of $g(X)$
- Consider the following examples:

$$X \sim U(0,1), \quad Y = X^n \sim ???$$

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$$X \sim f_X, \quad Y = X^2 \sim ???$$

Example 8

Let $X \sim U(0,1)$. Find the distribution of $Y = X^n$.

Proof.

For $0 \leqslant y \leqslant 1$,

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leqslant y) \\ &= \mathbb{P}(X^n \leqslant y) \\ &= \mathbb{P}(X \leqslant y^{1/n}) \\ &= F_X(y^{1/n}) \\ &= y^{1/n}. \end{aligned}$$

Therefore, the pdf is given by

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \begin{cases} \frac{1}{n} y^{1/n-1} & 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

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Monotone functions

Proposition 9

Let X be a continuous random variable having probability density function f_X. Suppose that $g(x)$ is a strictly monotone (increasing or decreasing), differentiable function of x. Then the random variable $Y = g(X)$ has a probability density function given by

$$f_Y(y) = \begin{cases} f_X[g^{-1}(y)]\left|\frac{d}{dy}g^{-1}(y)\right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

where $g^{-1}(y)$ is the inverse function of g.

Example 10 (The lognormal distribution)

If $X \sim N(0,1)$, find the distribution of $Y = e^X$.

Solution.

Note that $g(x) = e^x$ is an increasing function, and

$$g^{-1}(y) = \log y, \quad \frac{d}{dy}g^{-1}(y) = \frac{1}{y}, \quad y > 0.$$

Then, the pdf of Y is given by

$$f_Y(y) = f_X[g^{-1}(y)]\left|\frac{d}{dy}g^{-1}(y)\right| = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}(\log y)^2\right\} \cdot \frac{1}{y}, \quad y > 0.$$

and equals to 0 otherwise. ■

Example

Example 11

If X is a continuous random variable with probability density f_X, find the distribution of $Y = X^2$.

Theorem 12

Let X be a random variable with cdf $F(x)$. Assume that F is strictly increasing and continuous, then

$$Y := F(X) \sim U(0, 1).$$

Proof.

In order to find the distribution of Y, it suffices to find the cdf of Y: $F_Y(y) = \mathbb{P}(Y \leqslant y)$ for $y \in \mathbb{R}$.
We only consider the case where $y \in [0,1]$. (Why?)
If $0 \leqslant y \leqslant 1$, then

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leqslant y) = \mathbb{P}(F(X) \leqslant y) \\ &= \mathbb{P}(X \leqslant F^{-1}(y)) = F(F^{-1}(y)) = y. \end{aligned}$$

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Inverse transformation

Inverse function $F^{-1}(\cdot)$

- The function F may be flat...
- and thus F^{-1} does not always exits, that is,

$$F^{-1}(\{u\}) \quad \text{may not be unique.}$$

- Choose a respective from the points as its inverse.

Definition 13

If X has the distribution function $F(x)$. For any u, define

$$F^{-1}(u) = \inf\{x : F(x) \geqslant u\}.$$

Theorem 14

Let F be an arbitrary distribution function, and let $U \sim U(0,1)$. Then,

$$X = F^{-1}(U)$$

has the distribution function F.

Example 15

Note that the distribution function of $\mathrm{Exp}(\lambda)$ is

$$F(x) = 1 - e^{-\lambda x}.$$

Then,

$$F^{-1}(u) = -\frac{1}{\lambda}\log(1-u).$$

Therefore, if $U \sim U(0,1)$, then

$$F^{-1}(U) \sim \mathrm{Exp}(\lambda).$$

Example

Example 16

Assume that X is a discrete random variable with the pmf:

x	1	2	3	4
$p(x)$	0.1	0.2	0.3	0.4

Then, the distribution function of X is

$$F(x) = \begin{cases} 0 & x < 1 \\ 0.1 & 1 \leqslant x < 2 \\ 0.3 & 2 \leqslant x < 3 \\ 0.6 & 3 \leqslant x < 4 \\ 1 & x \geqslant 4 \end{cases}$$

Find $F^{-1}(u)$ for $0 < u < 1$.