一些补充的材料

Foundation of Probability Theory/STA 203

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Fall, 2023

Monotone property

Unions and intersections

Let $\mathscr{J}$ be a finite or countable index set. Let $\{A_j, j \in \mathscr{J}\}$ be a family of sets.

$$\bigcup_{j \in \mathscr{J}} A_j = \{x : x \in A_j \text{ for some } j \in \mathscr{J}\},$$

$$\bigcap_{j \in \mathscr{J}} A_j = \{x : x \in A_j \text{ for all } j \in \mathscr{J}\}.$$

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Example 1

Consider the following collection of sets indexed by $\mathbb{N}$:

$$A_1 = (0, 1), \quad A_2 = (0, \frac{1}{2}), \quad A_3 = (0, \frac{1}{3}), \quad \ldots, A_n = (0, \frac{1}{n}), \ldots$$

Show that

(i) $\bigcup_{n=1}^{\infty} A_n = (0, 1)$;

(ii) $\bigcap_{n=1}^{\infty} A_n = \varnothing$.

Solution.

(i) Let $x \in \bigcup_{n=1}^{\infty} A_n$, then $x \in A_n = (0, \frac{1}{n})$ for some $n \geqslant 1$, which further implies that $x \in (0, 1)$. This shows that

$$\bigcup_{n=1}^{\infty} A_n \subset (0, 1).$$

For the other side,

$$(0, 1) = A_1 \subset \bigcup_{n=1}^{\infty} A_n.$$

Therefore, (i) is proved.

(ii) By contradiction.

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Examples

Example 2

Prove that $\bigcup_{n=1}^{\infty}(0, \frac{n}{n+1}] = (0, 1)$.

Proof.

(i) Step 1: $\bigcup_{n=1}^{\infty}(0, \frac{n}{n+1}] \subset (0,1)$. Let $x \in \bigcup_{n=1}^{\infty}(0, \frac{n}{n+1}]$, then $x \in (0, \frac{n}{n+1}]$ for some $n \geqslant 1$. Thus, $0 \leqslant x \leqslant \frac{n}{n+1} < 1$, which implies that $x \in (0,1)$.

(ii) Step 2: $(0,1) \subset \bigcup_{n=1}^{\infty}(0, \frac{n}{n+1}]$. Let $x \in (0,1)$, and define $\varepsilon = 1 - x > 0$. Then, there exists a number N such that

$$\varepsilon > \left|\frac{N}{N-1} - 1\right|.$$

Therefore,

$$1 - x = \varepsilon > 1 - \frac{N}{N+1} \quad \Longrightarrow \quad x < \frac{N}{N+1}.$$

Hence,

$$x \in (0, \frac{N}{N+1}] \in \bigcup_{n=1}^{N}(0, \frac{n}{n+1}] \in \bigcup_{n=1}^{\infty}(0, \frac{n}{n+1}].$$

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Example 3

Show that

$$\bigcap_{n=1}^{\infty}(1-\frac{1}{n},3] = [1,3].$$

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Definition 4

Let Ω be a sample space. $\mathscr{F}$ is a σ-field if

(i) $\Omega \in \mathscr{F}$;

(ii) If $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$;

(iii) If $A_1, A_2, \cdots \in \mathscr{F}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}.$$

Proposition 5

(i) $\varnothing \in \mathscr{F}$;

(ii) If $A_1, A_2, \cdots \in \mathscr{F}$, then

$$\bigcap_{n=1}^{\infty} A_n \in \mathscr{F}.$$

(iii) If $A_1, A_2, \ldots, A_n \in \mathscr{F}$, then

$$\bigcup_{i=1}^{n} A_i \in \mathscr{F}.$$

Borel set in $\mathbb{R}$

Definition 6

The Borel set in $\mathbb{R}$, denoted by $\mathscr{B}(\mathbb{R})$, is defined as the smallest σ-field containing all intervals $(a, b]$ where $a < b \in \mathbb{R}$.

Proposition 7

(i) For any $x \in \mathbb{R}$,

$$\{x\} \in \mathscr{B}(\mathbb{R}).$$

(ii) For any $x < y \in \mathbb{R}$,

$$(x, y), [x, y), [x, y], (-\infty, y], (x, \infty) \in \mathscr{B}(\mathbb{R}).$$

Increasing events

In the context of probability, increasing events are events that become more likely to occur as additional information is given.

Definition 8

A sequence of events $\{E_n, n \geqslant 1\}$ is said to be an increasing sequence if

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset E_{n+1} \subset \ldots .$$

Example 9

An example of increasing events can be rolling a fair six-sided die:

- Event E_1: The outcome is less than or equal to 3.
- Event E_2: The outcome is less than or equal to 4.
- Event E_3: The outcome is less than or equal to 5.

Limit of increasing events

Definition 10

If $\{E_n, n \geqslant 1\}$ is an increasing sequence of events, then $\lim_{n\to\infty} E_n$ is defined by

$$\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i.$$

The following proposition is the so called monotone property:

Proposition 11

If $\{E_n, n \geqslant 1\}$ is an increasing sequence of events with $E_\infty = \lim_{n\to\infty} E_n$, then

$$\mathbb{P}\big(E_\infty\big) = \lim_{n\to\infty} \mathbb{P}(E_n).$$

Proof.

Define the events F_n for $n \geqslant 1$ by

$$F_1 = E_1, \quad F_2 = E_2 \setminus E_1, \quad \ldots, \quad F_n = E_n \setminus E_{n-1}, \quad \ldots.$$

In words, F_n consists of those outcomes in E_n which are not in any of the earlier $E_j, j < n$. It is easy to verify that F_n are mutually exclusive events such that

$$\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i = E_n, \quad \text{for all } n \geqslant 1 \text{ and } n = \infty.$$

Then,

$$\begin{aligned}\mathbb{P}(E_\infty) &= \mathbb{P}\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(F_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(F_i) = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{n} F_i\right) \\ &= \lim_{n\to\infty} \mathbb{P}(E_n).\end{aligned}$$

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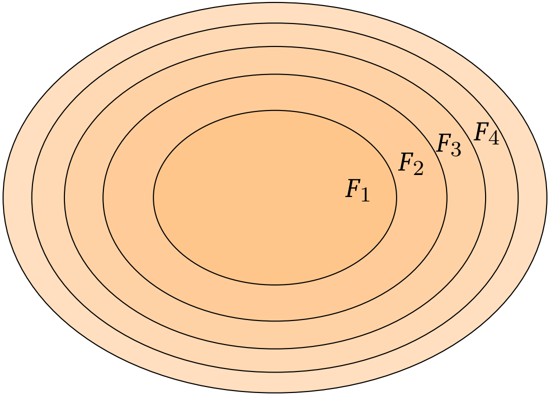


F_4
F_3
F_2
F_1

Decreasing events

Definition 12

A sequence $\{E_n, n \geqslant 1\}$ is said to be a decreasing seqeunce if $E_1 \supset E_2 \supset \dots$. Its limit is defined by

$$\lim_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} E_n.$$

Proposition 13

If $\{E_n, n \geqslant 1\}$ is decreasing with $E_\infty = \lim_{n\to\infty} E_n$, then

$$\mathbb{P}(E_\infty) = \lim_{n\to\infty} \mathbb{P}(E_n).$$

Axioms of continuity

Proposition 14 (Axioms of continuity)

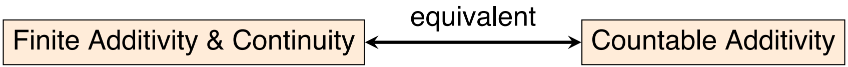
If $E_n \downarrow \varnothing$, then $\mathbb{P}(E_n) \to 0$ as $n \to \infty$.

Remark

This proposition is a special case of the monotone property.

Theorem 15

The axioms of finite additivity and continuity together are equivalent to the axiom of countable additivity.



Proof.

Step 1. Proof of "Countable additivity" $\Longrightarrow$ "Finite Additivity & Continuity". Proved.

Step 2. Proof of "Finite Additivity & Continuity" $\Longrightarrow$ "Countable additivity". Let $\{E_n, n \geqslant 1\}$ be pairwise disjoint, then $F_n := \cup_{k=n+1}^{\infty} E_k \downarrow \emptyset$. By the "Continuity" property, $\lim_{n\to\infty} \mathbb{P}(F_n) = 0$. If "Finite additivity" is assumed, then

$$1 \geqslant \mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) + \mathbb{P}(F_n) = \sum_{i=1}^{n} \mathbb{P}(E_i) + \mathbb{P}(F_n).$$

Let $a_n = \sum_{i=1}^{n} \mathbb{P}(E_i)$. It follows that $a_n \uparrow$ and bounded by 1 (why?), and thus the limit $\lim_{n\to\infty} a_n$ exists. Taking limits on both sides yields

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} \mathbb{P}(F_n) = \sum_{i=1}^{\infty} \mathbb{P}(E_i).$$

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Example: Tossing a Coin

Example 16 (Problem Statement)

Consider an experiment where a fair coin is tossed until the first head appears. Let A_i be the event that the first head appears on or before the i-th toss. As i increases, A_i forms an increasing sequence of events.

Application of Continuity Property

According to the first continuity property, we can say that the probability of getting a head eventually is the limit of the probabilities of A_i as i goes to infinity, i.e.,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mathbb{P}(A_i) = \lim_{i\to\infty} (1 - 2^{-i}) = 1.$$

Probability zero sets

- $\varnothing$ has probability 0, but the inverse is not correct:
- Not all of probability zero sets are empty.
- For example, in the probability space $(\mathscr{U}, \mathscr{B}, m)$,

$$m(\{0.5\}) = m\left(\bigcap_{n=1}^{\infty}(0.5 - \frac{1}{2n}, 0.5]\right) = \lim_{n\to\infty} m((0.5 - \frac{1}{2n}, 0.5]) = 0.$$

- Intuitively, the set $\{0.5\}$ has length 0, and then the probability of $\{0.5\}$ is 0.
- As a result,

$$m([a, b]) = m((a, b)),$$

because $m(\{a\}) = m(\{b\}) = 0$.

A formal definition of probability zero sets

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Definition 17

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A set $E \in \mathscr{F}$ is said to have probability zero if for any $\varepsilon > 0$, there exists a countable number of subsets E_n such that $E \subset \cup_{n=1}^{\infty} E_n$, and

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \varepsilon.$$

Example 18 (The rational number set has probability zero)

In the probability space $(\mathscr{U}, \mathscr{B}, m)$, let $E = \mathbb{Q} \cap (0, 1]$ be the collection of all rational number in $\mathscr{U} = (0, 1]$. Then, $\mathbb{P}(E) = 0$.

Almost surely

When we make probabilistic claims without considering the measure zero sets, we say that an event happens almost surely.

Definition 19 (Almost surely)

An event E is said to hold almost surely (a.s.) if $\mathbb{P}(E) = 1$.

Example 20 (Irrational numbers)

In the probability space $(\mathscr{U}, \mathscr{B}, m)$, let E be the event containing all of the irrational numbers. Then

$$\mathbb{P}(E) = 1.$$

Problems

1. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ then

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \inf_{i \geqslant 1} \mathbb{P}(A_i)$$

A. True

B. False

Problems

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2. If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$, then

$$\mathbb{P}(\bigcap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mathbb{P}(A_i)$$

A. True

B. False

Further reading

[1] Sheldon M. Ross (谢尔登·M. 罗斯).

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[3] Kai-Lai Chung (钟开莱).

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3rd edition (原书第三版), 机械工业出版社

Further reading

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2nd Edition. MIT.

[5] Stanley H. Chan.

Introduction to Probability for Data Science.

Michigan Publishing. (FREE on website)