Lecture note 8: Multivariate normal distribution

Foundation of Probability Theory/STA 203

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Bivariate normal distribution

Introduction

The bivariate normal distribution is commonly used to model the joint distribution of two random variables with a linear relationship. Here are some real-world examples where the bivariate normal distribution might be a reasonable model:

- Height and weight of adults
- Father and son's heights
- Test scores in two subjects
- We can assume that these variables both have a marginal normal distribution.
- However, there are some correlation between them.

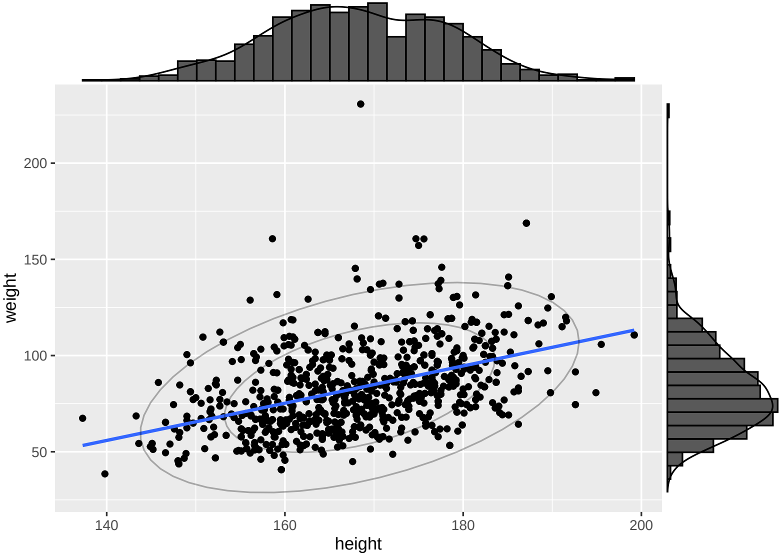


Figure: This is survey data collected by the US National Center for Health Statistics (NCHS)

- Let X be the height (in cm), and Y be the weight (in kg).
- What can you see from the marginal distributions of X and Y?
- Are they independent?

老忠实间歇泉（英语：Old Faithful）是一座位于美国黄石国家公园的间歇泉，为黄石国家公园第一个被命名的间歇泉。现喷发规律是80 分钟左右一次。

Let X be the waiting time (in minutes), and let Y be the duration time of the eruptions (in minutes). History data gives the following graph:

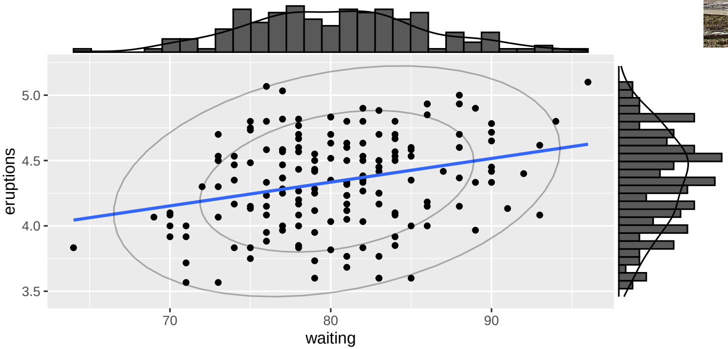


Figure: Old Faithful

Questions

- Whether X and Y are independent? Are they correlated?
- What is the joint pdf of X and Y? How about marginal pdfs?
- What is the conditional distribution of Y given that $X = 80$? How about the conditional expectation?

Joint pdf of independent normal variables

- If $X \sim N(0,1)$ and $Y \sim N(0,1)$ are independent, then

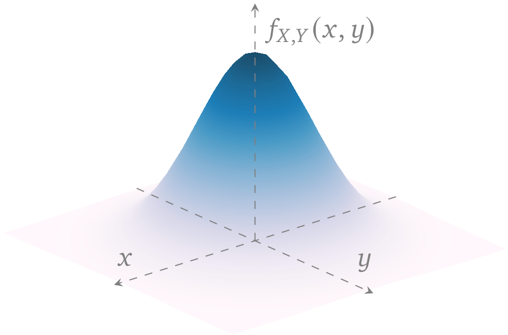
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

- Then the joint pdf is

$$f(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi} e^{-\frac{x^2}{2} - \frac{y^2}{2}}.$$

- We say X and Y follow a standard bivariate normal distribution.

A figure



Bivariate Normal Distribution

The bivariate normal distribution is a probability distribution that describes the joint distribution of two normally distributed variables.

Definition 1

X and Y are said to be bivariate normally distributed with means μ_X and μ_Y and variances σ_X^2 and σ_Y^2 respectively, and with correlation coefficient ρ, if the joint pdf of X and Y is given by:

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right)$$

Specially, if $\mu_X = \mu_Y = \rho = 0$ and $\sigma_X = \sigma_Y = 1$, then it is said to be a standard bivariate normal distribution.

Z-scores

Definition 2

The z-score of a random variable X is defined as

$$X^* = \frac{X - \mu}{\sigma},$$

where $\mu = \mathbb{E}[X]$ and $\sigma^2 = \mathrm{Var}(X)$.

Remark

It can be shown that if $X \sim N(\mu, \sigma^2)$, then

$$X^* = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

Distribution of the Z scores

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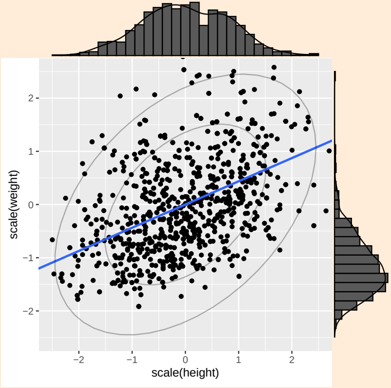
Proposition 3

If X and Y follow the bivariate normal distribution with parameters $(\mu_X, \mu_Y; \sigma_X^2, \sigma_Y^2, \rho)$, then X^* and Y^* follow the bivariate normal distribution with parameters $(0, 0; 1, 1, \rho)$, where

$$\rho = \mathrm{Cor}(X, Y) = \mathrm{Cor}(X^*, Y^*) = \mathrm{Cov}(X^*, Y^*).$$



Proof.

Let

$$x^* = g(x, y) = \frac{x - \mu_X}{\sigma_X}, \quad y^* = h(x, y) = \frac{y - \mu_Y}{\sigma_Y},$$

then

$$|J| = \begin{vmatrix} \frac{1}{\sigma_X} & 0 \\ 0 & \frac{1}{\sigma_Y} \end{vmatrix} = \frac{1}{\sigma_X \sigma_Y}.$$

Then, it follows that

$$f_{X^*,Y^*}(x^*, y^*) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[(x^*)^2 - 2\rho x^* y^* + (y^*)^2\right]\right). \quad \blacksquare$$

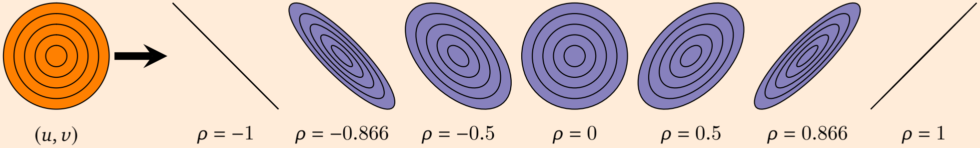
Linear transformations

Proposition 4

If $U \sim N(0,1), V \sim N(0,1)$ are independent random variables and $\rho \in [-1,1]$, and let

$$X = U, \quad Y = \rho U + \sqrt{1-\rho^2}V,$$

then (X,Y) follows the bivariate normal distribution with parameters $(0,0;1,1,\rho)$.



Proof.

Note that the joint pdf of U and V is

$$f_{U,V}(u,v) = \frac{1}{2\pi} e^{-\frac{u^2}{2} - \frac{v^2}{2}},$$

and

$$u = x, \quad v = \frac{1}{\sqrt{1-\rho^2}}(y - \rho x),$$

The Jacobian determinant is given by

$$|J| = \begin{vmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{vmatrix} = \sqrt{1-\rho^2},$$

and thus, the joint pdf of X and Y is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2}{2} - \frac{(y-\rho x)^2}{2(1-\rho^2)}\right) \quad \blacksquare$$

Marginal distributions

Proposition 5

If X and Y follow the bivariate normal distribution with parameters $(\mu_X, \mu_Y; \sigma_X^2, \sigma_Y^2, \rho)$, then the marginal distributions of X and Y are given by

$$X \sim N(\mu_X, \sigma_X^2), \quad \text{and} \quad Y \sim N(\mu_Y, \sigma_Y^2),$$

respectively.

Remark

It follows that

$$\mathbb{E}[X] = \mu_X, \quad \mathrm{Var}(X) = \sigma_X^2, \quad \mathbb{E}[Y] = \mu_Y, \quad \mathrm{Var}(Y) = \sigma_Y^2.$$

As $\mathrm{Cor}(X, Y) = \rho$, we have

$$\mathrm{Cov}(X, Y) = \rho\sigma_X\sigma_Y.$$

Proof.

We only show for the case where $\mu_X = \mu_Y = 0$ and $\sigma_X = \sigma_Y = 1$. In this case,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2}{2} - \frac{(y-\rho x)^2}{2(1-\rho^2)}\right) dy \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right) dy \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \end{aligned}$$

■

Conditional distribution

Proposition 6

If X and Y follow the bivariate normal distribution with parameters $(\mu_X, \mu_Y; \sigma_X^2, \sigma_Y^2, \rho)$, then

$$Y|X = x \sim N\left(\mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(x - \mu_X), (1-\rho^2)\sigma_Y^2\right),$$

or, equivalently,

$$Y^*|X^* = x^* \sim N(\rho x^*, 1-\rho^2),$$

where (X^*, Y^*) is the z-score of (X, Y).
Moreover,

$$\mathbb{E}[Y|X = x] = \mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(x - \mu_X), \quad \mathrm{Var}(Y|X = x) = (1-\rho^2)\sigma_Y^2.$$

Linear regression

- Usually, the joint distribution of (X, Y) is unknown.
- The regression function

$$h(x) = \mathbb{E}[Y|X = x]$$

  is also unknown.
- However, if we assume that (X, Y) is a bivariate normal random vector, then

$$h(x) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = b_0 + b_1 x,$$

  where

$$b_0 = \mu_Y - b_1\mu_X, \quad b_1 = \rho\frac{\sigma_Y}{\sigma_X}.$$

Examples

Example 7

Assume that the height and weight of a randomly chosen adult, X and Y, follow a bivariate normal distribution with parameters

$$\mu_X = 168.84, \mu_Y = 82.05; \sigma_X^2 = 101.74, \sigma_Y^2 = 448.84, \rho = 0.45.$$

Find (a) $\mathbb{P}\{160 < X < 180\}$. (b) $\mathbb{E}[Y|X = 170]$. (c) $\mathrm{Var}(Y|X = 180)$.

Examples

Solution.

(a) As $X \sim N(168.84, 101.74)$, then $X^* = \frac{X-168.84}{\sqrt{101.74}} \sim N(0,1)$, and hence

$$\mathbb{P}\{160 < X < 180\} = \mathbb{P}\left\{\frac{160-168.84}{\sqrt{101.74}} < X^* < \frac{180-168.84}{\sqrt{101.74}}\right\}$$
$$= \mathbb{P}\{-0.876 < X^* < 1.106\} \approx 0.675.$$

Examples

(b) We have

$$\mathbb{E}[Y|X = 170] = 82.05 + \frac{(0.45)(\sqrt{448.84})}{\sqrt{101.74}}(170 - 168.84) = 83.147.$$

(c) We have

$$\mathrm{Var}[Y|X = 180] = (1 - 0.45^2)(448.84) = 358.28.$$

Actually, we can also obtain

$$Y|X = 170 \sim N(83.147, 358.28).$$

■

Properties

Proposition 8 (Independence)

If X and Y are bivariate normal and uncorrelated, then they are independent.

Example 9

If X and Y follow the bivariate normal distribution with parameters $(\mu_X, \mu_Y; \sigma_X^2, \sigma_Y^2, \rho)$, find the joint distribution of X and $W = Y - \frac{\rho\sigma_Y}{\sigma_X}X$. Whether they are independent?

Properties

Proposition 10 (Linear combinations of X and Y)

Random variables X and Y follow the bivariate normal distribution with parameters $(\mu_X, \mu_Y; \sigma_X^2, \sigma_Y^2, \rho)$, if and only if for any $a, b \in \mathbb{R}$,

$$aX + bY \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + 2ab\rho\sigma_X\sigma_Y + b^2\sigma_Y^2).$$

Example 11

Let X and Y be jointly normal random variables with parameters $\mu_X = 1, \sigma_X^2 = 1, \mu_Y = 0, \sigma_Y^2 = 4$, and $\rho = 1/2$. Find (a) $\mathbb{P}\{2X + Y \leqslant 3\}$, (b) $\mathrm{Cov}(X + Y, 2X - Y)$, and (c) $\mathbb{P}\{Y > 1 | X = 2\}$.

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Solution.

(a) Since X and Y are jointly normal, then $2X+Y \sim N(2\mu_X+\mu_Y, 4\sigma_X^2+2\rho(2\sigma_X)\sigma_Y+\sigma_Y^2) = N(2,12)$. Therefore,

$$\mathbb{P}\{V \leqslant 3\} = \mathbb{P}\left\{V^* \leqslant \frac{3-2}{\sqrt{12}}\right\} \approx \Phi(0.2887) \approx 0.6136.$$

(b) Note that $\mathrm{Cov}(X,Y) = \rho\sigma_X\sigma_Y = 1$. Therefore,

$$\mathrm{Cov}(X+Y, 2X-Y) = 2\,\mathrm{Var}(X) + 2\,\mathrm{Cov}(X,Y) - \mathrm{Cov}(X,Y) - \mathrm{Var}(Y) = -1.$$

Examples

Solution (Cont'd).

(c) As

$$\mathbb{E}[Y|X=2] = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(2-\mu_X) = 1, \quad \mathrm{Var}(Y|X=2) = (1-\rho^2)\sigma_Y^2 = 3,$$

it follows that $Y|X=2 \sim N(1,3)$, and therefore,

$$\mathbb{P}\{Y > 1|X=2\} = 1 - \Phi\left(\frac{1-1}{\sqrt{3}}\right) = 0.5.$$

Notice!

If X and Y are jointly normal, then each random variable X and Y is normal. However, the converse is not true.

Example 12

Let $X \sim N(0, 1)$ and

$$W = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

be independent random variables. Let $Y = WX$. Find the pdf of Y. Does (X, Y) bivariate normal distributed? Why? Or why not?

Solution.

By symmetry of $N(0,1)$, we have $-X \sim N(0,1)$. Therefore,

$$\begin{aligned}\mathbb{P}\{Y \leqslant y\} &= \mathbb{P}\{Y \leqslant y | W = -1\}\,\mathbb{P}\{W = -1\} + \mathbb{P}\{Y \leqslant y | W = 1\}\,\mathbb{P}\{W = 1\} \\ &= \frac{1}{2}\mathbb{P}\{X \leqslant y\} + \frac{1}{2}\mathbb{P}\{-X \leqslant y\} \\ &= \frac{1}{2}\Phi(y) + \frac{1}{2}\Phi(y) = \Phi(y).\end{aligned}$$

Hence, $Y \sim N(0,1)$.

However, X and Y are not jointly normal, because $Z = X + Y$ has the following form:

$$Z = \begin{cases} 2X & \text{if } W = 1 \\ 0 & \text{if } W = -1. \end{cases}$$

Therefore, if $z \geqslant 0$,

$$\begin{aligned}\mathbb{P}\{Z \leqslant z\} &= \mathbb{P}\{Z \leqslant z | W = 1\}\,\mathbb{P}\{W = 1\} + \mathbb{P}\{Z \leqslant z | W = -1\}\,\mathbb{P}\{W = -1\} \\ &= \frac{1}{2}\,\mathbb{P}\{X \leqslant \frac{z}{2}\} + \frac{1}{2} = \frac{1}{2}(1 + \Phi(\frac{z}{2})),\end{aligned}$$

while if $z < 0$,

$$\mathbb{P}\{Z \leqslant z\} = \frac{1}{2}\,\mathbb{P}\{X \leqslant \frac{z}{2}\} = \frac{1}{2}\Phi(\frac{z}{2}).$$

This example illustrates that although X and Y are normally distributed, it is possible that their sum Z is not normally distributed, which further implies that X and Y are not jointly normal. ■

Properties

Some important properties of the bivariate normal distribution include:

- The marginal distributions of X and Y are themselves normally distributed.
- The conditional distribution of X given $Y = y$ and the conditional distribution of Y given $X = x$ are both normally distributed with means and variances that depend on y and x respectively.
- The conditional expectation of X given $Y = y$ and the conditional expectation of Y given $X = x$ are both linear functions of y and x respectively.

Multivariate normal distribution

Multivariate Normal Distribution

The multivariate normal distribution is a probability distribution that describes the joint distribution of p normally distributed variables.

Definition 13

If $\boldsymbol{X} = (X_1, X_2, \dots, X_p)$ is a p-dimensional random vector with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, then the pdf of multivariate normal distribution is given by:

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right),$$

and we denote $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Here,

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix} \in \mathbb{R}^p, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \dots & \sigma_{1p} \\ \vdots & \ddots & \vdots \\ \sigma_{p1} & \dots & \sigma_{pp} \end{pmatrix} \in \mathbb{R}^{p\times p},$$

and $\boldsymbol{\Sigma}$ is a positive definite matrix. The symbol $|\boldsymbol{\Sigma}|$ is the determinant of $\boldsymbol{\Sigma}$.

Standard MND

Definition 14

Specially, if $\boldsymbol{\mu} = \mathbf{0}$, and $\boldsymbol{\Sigma} = \boldsymbol{I}_p$, then we say $\boldsymbol{X}$ follows a standard multivariate normal distribution if $\boldsymbol{X} \sim N(\mathbf{0}, \boldsymbol{I}_p)$.

Properties

Some important properties of the multivariate normal distribution include:

- Any linear combination of the components of $\boldsymbol{X}$ is also normally distributed.
- The marginal distributions of any subset of components of $\boldsymbol{X}$ are themselves multivariate normal.
- The conditional distribution of any subset of components of $\boldsymbol{X}$ given the remaining components is also multivariate normal.
- The conditional expectation of any subset of components of $\boldsymbol{X}$ given the remaining components is a linear function of the remaining components.

Properties of $f_{\boldsymbol{X}}(\boldsymbol{x})$

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Proposition 15

We have

$$\int_{\mathbb{R}^p} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} = 1.$$

Proof.

Since $\boldsymbol{\Sigma} > 0$, it follows that a non-singular matrix $\boldsymbol{L}$ such that

$$\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T, \quad |\boldsymbol{L}| = |\boldsymbol{\Sigma}|^{1/2}.$$

Consider the transformation

$$\boldsymbol{y} = \boldsymbol{L}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}).$$

Then,

$$\boldsymbol{x} = \boldsymbol{L}\boldsymbol{y} + \mu,$$

Therefore,

$$(\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}) = \boldsymbol{y}^T \boldsymbol{y}.$$

■

Moment generating function of $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

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Theorem 16

The moment generating function of $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is given by

$$M(\boldsymbol{t}) = \exp\left\{\boldsymbol{\mu}^T\boldsymbol{t} + \frac{1}{2}\boldsymbol{t}^T\boldsymbol{\Sigma}\boldsymbol{t}\right\}, \quad \boldsymbol{t} \in \mathbb{R}^p.$$

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Definition 17

For $\boldsymbol{\mu} \in \mathbb{R}^p$, and $\boldsymbol{\Sigma} \in \mathbb{R}^{p\times p}$ is a non-negative definite matrix. Then $\boldsymbol{X}$ is called to follow a multivariate normal distribution if its moment generating function is

$$M(\boldsymbol{t}) = \exp\left\{\boldsymbol{\mu}^T\boldsymbol{t} + \frac{1}{2}\boldsymbol{t}^T\boldsymbol{\Sigma}\boldsymbol{t}\right\}.$$

Remark

Here, $\boldsymbol{\Sigma}$ may be degenerate, say, rank$(\boldsymbol{\Sigma}) < p$, or $|\boldsymbol{\Sigma}| = 0$. In this case, we say $\boldsymbol{X}$ follows a degenerate normal distribution, or singular normal distribution.

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Theorem 18

Any subvector of $\boldsymbol{X}$, say,

$$\widetilde{X} = (X_{k_1}, \ldots, X_{k_r})^T, \quad r \leqslant p,$$

also follows a normal distribution $N(\widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\Sigma}})$, where

$$\widetilde{\boldsymbol{\mu}} = \begin{pmatrix} \mu_{k_1} \\ \vdots \\ \mu_{k_r} \end{pmatrix}, \quad \widetilde{\boldsymbol{\Sigma}} = \begin{pmatrix} \sigma_{k_1,k_1} & \ldots & \sigma_{k_1,k_r} \\ \vdots & \ddots & \vdots \\ \sigma_{k_r,k_1} & \ldots & \sigma_{k_r,k_r} \end{pmatrix}$$

Properties

Remark

The marginal distribution of X_j is $N(\mu_j, \sigma_{jj})$. The marginal distribution of (X_j, X_k) is

$$N\left(\begin{pmatrix}\mu_j \\ \mu_k\end{pmatrix}, \begin{pmatrix}\sigma_{jj} & \sigma_{jk} \\ \sigma_{jk} & \sigma_{kk}\end{pmatrix}\right).$$

Properties

Theorem 19

We have

$$\mu_j = \mathbb{E}[X_j], \quad \sigma_{jj} = \mathrm{Var}(X_j).$$

Moreover,

$$\sigma_{jk} = \mathrm{Cov}(X_j, X_k).$$

Independence

Theorem 20

Random variables $X_1, X_2, \ldots, X_p$ are independent, if and only if $\sigma_{jk} = 0$ for all $j \neq k$. Generally, if $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2)$, where $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are two subvectors of $\boldsymbol{X}$, and let

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix},$$

where $\boldsymbol{\Sigma}_{11}$ and $\boldsymbol{\Sigma}_{22}$ are the covariance matrices of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, respectively, and

$$\boldsymbol{\Sigma}_{12} = \mathbb{E}[(\boldsymbol{X}_1 - \boldsymbol{\mu}_1)(\boldsymbol{X}_2 - \boldsymbol{\mu}_2)^T].$$

Then, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent if and only if $\boldsymbol{\Sigma}_{12} = \mathbf{0}$.

Example 21

Assume that $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ follows $N\left(\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & 3 \end{pmatrix}\right)$.

Find

(a) The distributions of X_1, X_2 and X_3.

(b) The distribution of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$.

(c) Whether X_1 and X_2 are independent?

(d) Whether X_1 and $(X_2, X_3)^T$ are independent?

Linear transformation

- Let $\boldsymbol{X} \in \mathbb{R}^p$ be any random vector (not necessarily normal), satisfying

$$\mathbb{E}[\boldsymbol{X}] = \boldsymbol{\mu}, \mathrm{Cov}(\boldsymbol{X}) = \boldsymbol{\Sigma}.$$

- Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_p)^T$. Consider the linear transformation

$$Y = \sum_{j=1}^{p} a_j X_j = \boldsymbol{a}^T \boldsymbol{X}.$$

- It follows that

$$\mathbb{E}[Y] = \sum_{j=1}^{p} a_j \mu_j = \boldsymbol{a}^T \boldsymbol{\mu}.$$

- Moreover,

$$\mathrm{Var}(Y) = \sum_{j=1}^{p} \sum_{k=1}^{p} a_j a_k \sigma_{jk} = \boldsymbol{a}^T \boldsymbol{\Sigma} \boldsymbol{a}.$$

Linear transformation of $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

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Theorem 22

$\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ if and only if

$$\boldsymbol{a}^T \boldsymbol{X} \sim N\left(\sum_{j=1}^{p} a_j \mu_j, \sum_{j=1}^{p} \sum_{k=1}^{p} a_j a_k \sigma_{jk}\right) \quad \text{for any } \boldsymbol{a} \in \mathbb{R}^p.$$

Property of transformation of $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

Theorem 23

If $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then for any $\boldsymbol{C} \in \mathbb{R}^{r\times p}$,

$$\boldsymbol{C}\boldsymbol{X} \sim N(\boldsymbol{C}\boldsymbol{\mu}, \boldsymbol{C}\boldsymbol{\Sigma}\boldsymbol{C}^T).$$

Theorem 24

If $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then there exists a orthogonal transformation $\boldsymbol{U}$ such that each component of $\boldsymbol{UX}$ is independent of each other. More specifically,

$$\boldsymbol{UX} \sim N(\boldsymbol{U\mu}, \boldsymbol{\Lambda}),$$

where

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_p \end{pmatrix}$$

and λ_j's are the eigenvalues of $\boldsymbol{\Sigma}$.

Example 25

Assume that $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ follows $N\left[\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\right]$.

Find

(a) the distribution of $X_1 - 2X_2 + X_3$;

(b) the joint distribution of $X_1 - X_2 + X_3$ and $3X_1 + X_2 - 2X_3$;

(c) an orthogonal matrix $\boldsymbol{U}$ such that $\boldsymbol{U}\boldsymbol{X}$ has independent components.

Solution.

(a) Let $Y = \boldsymbol{a}^T\boldsymbol{X}$, where $\boldsymbol{a} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, then $Y = X_1 - 2X_2 + X_3$. Note that

$$\boldsymbol{a}^T\boldsymbol{\mu} = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 3, \quad \boldsymbol{a}^T\boldsymbol{\Sigma}\boldsymbol{a} = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 15.$$

The distribution of Y is $N(3, 15)$.

Solution

(b) Let $\boldsymbol{a}_1 = (1, -1, 1)^T$ and $\boldsymbol{a}_2 = (3, 1, -2)$, and let

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 1 \\ 3 & 1 & -2 \end{pmatrix}$$

Then,

$$\boldsymbol{AX} = \begin{pmatrix} X_1 - 2X_2 + X_3 \\ 3X_1 + X_2 - 2X_3 \end{pmatrix}$$

Note that

$$\boldsymbol{A\mu} = \begin{pmatrix} 1 & -1 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix},$$

$$\boldsymbol{A\Sigma A}^T = \begin{pmatrix} 1 & -1 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 6 & 45 \end{pmatrix}.$$

Solution.

(c) The eigenvalues of $\boldsymbol{\Sigma}$ are $4, 2$ and 0, and the eigenvectors are

$$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix},$$

Then, with

$$\boldsymbol{U} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)^T, \boldsymbol{\Lambda} = \text{diag}(4, 2, 0).$$

we have $\boldsymbol{\Sigma} = \boldsymbol{U}^T \boldsymbol{\Lambda} \boldsymbol{U}$. As a consequence,

$$\text{Cov}(\boldsymbol{U}\boldsymbol{X}) = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^T = \boldsymbol{\Lambda}.$$

∎

Chi-squared distribution

Theorem 26

If $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $|\boldsymbol{\Sigma}| > 0$, then

$$(\boldsymbol{X} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{X} - \boldsymbol{\mu}) \sim \chi_p^2.$$

Conditional distribution

Theorem 27

If $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{pmatrix}$ follows a p-variate normal distribution $N(\begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix})$, then

$$\boldsymbol{X}_2 | \boldsymbol{X}_1 \sim N(\boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{X}_1 - \boldsymbol{\mu}_1), \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}).$$

Example

Example 28

Let

$$\boldsymbol{X} \sim N\left[\begin{pmatrix} 2 \\ 5 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 & 0 & 3 & 3 \\ 0 & 1 & -1 & 2 \\ 3 & -1 & 6 & -3 \\ 3 & 2 & -3 & 7 \end{pmatrix}\right].$$

Let

$$\boldsymbol{Y} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \boldsymbol{Z} = \begin{pmatrix} X_3 \\ X_4 \end{pmatrix}$$

Find the distribution of $\boldsymbol{Y}|\boldsymbol{Z} = \boldsymbol{z}$.

Solution.

Note that

$$\boldsymbol{\mu}_Y = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{\mu}_Z = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \boldsymbol{\Sigma}_{YY} = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\Sigma}_{ZZ} = \begin{pmatrix} 6 & -3 \\ -3 & 7 \end{pmatrix}, \quad \boldsymbol{\Sigma}_{YZ} = \begin{pmatrix} 3 & 3 \\ -1 & 2 \end{pmatrix} = \boldsymbol{\Sigma}_{ZY}^T.$$

Then,

$$\begin{aligned}
\mathbb{E}[\boldsymbol{Y}|\boldsymbol{Z} = \boldsymbol{z}] &= \boldsymbol{\mu}_Y + \boldsymbol{\Sigma}_{YZ}\boldsymbol{\Sigma}_{ZZ}^{-1}(\boldsymbol{z} - \boldsymbol{\mu}_X) \\
&= \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 6 & -3 \\ -3 & 7 \end{pmatrix}^{-1}\begin{pmatrix} z_1 + 2 \\ z_2 - 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 + \frac{10}{11}z_1 + \frac{9}{11}z_2 \\ \frac{14}{3} - \frac{1}{33}z_1 + \frac{3}{11}z_2 \end{pmatrix}.
\end{aligned}$$

Solution

Moreover,

$$\begin{aligned}\mathrm{Cov}(\boldsymbol{Y}|\boldsymbol{Z}=\boldsymbol{z}) &= \boldsymbol{\Sigma}_{YY} - \boldsymbol{\Sigma}_{YZ}\boldsymbol{\Sigma}_{ZZ}^{-1}\boldsymbol{\Sigma}_{ZY} \\ &= \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 6 & -3 \\ -3 & 7 \end{pmatrix}^{-1}\begin{pmatrix} 3 & -1 \\ 3 & 2 \end{pmatrix} \\ &= \frac{1}{33}\begin{pmatrix} 126 & -24 \\ -24 & 14 \end{pmatrix}.\end{aligned}$$

Fisher's Lemma

Theorem 29

Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ variables. Let

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \hat{\sigma}_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Then,

(i) $\bar{X}$ and $\hat{\sigma}_n^2$ are independent;

(ii) $\bar{X} \sim N(\mu, \sigma^2/n)$;

(iii) $(n-1)\hat{\sigma}_n^2/\sigma^2 \sim \chi_{n-1}^2$.

Further reading

[1] Sheldon M. Ross (谢尔登·M. 罗斯).

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[2] 李贤平.

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